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Mh4714 Week 1

## Week 1

### 0.1 Introduction

## NOTATION

Curly brackets $\{\ldots\}$ will be used to denote a set.

## Example 0.1

$\{1, a, 0,5,-\sqrt{2}\}$ denotes the set with elements $1, a, 0,5$ and $-\sqrt{2}$.
$\{1,2,3, \ldots\}$ could denote the set of integers greater than 0 .
$\{\cdots-3,-2,-1,0,1,2,3, \ldots\}$ could denote set of all integers.
$\{1,3,5,7, \ldots\}$ could denote the set of positive odd numbers.
$\{1,4,9,16, \ldots\}$ could denote the set of square numbers.

A set is completely determined by its elements.

## Example 0.2

$$
\{1,2,3\}=\{3,1,2\}=\{1,1,2,3\}
$$

## Special Symbols:

$\varnothing$ denotes the empty or null set.
$\mathbb{N}$ denotes the set of integers greater than 0 denoted above as $\{1,2,3, \ldots\}$.
$\mathbb{Z}$ denotes the set of all integers denoted above as $\{\cdots-3,-2,-1,0,1,2,3, \ldots\}$.
$\mathbb{Q}$ denotes the set of rationals, that is, the set of all numbers of the form $\frac{a}{b}$
where $a \in \mathbb{Z}, b \in \mathbb{Z} \backslash\{0\}$.
$\mathbb{R}$ denotes the set of real numbers and consists of all rational numbers together with all irrational numbers such as $\sqrt{2}, \sqrt{3}, \sqrt[3]{2}, \pi, e$.

## Special Notation:

$\{2 n: n \in \mathbb{N}\}$ denotes the set all even numbers.
$\left\{\frac{a}{b}: a \in \mathbb{Z}, b \in \mathbb{Z} \backslash\{0\}\right\}$ denotes the set of rational numbers.
$\{x \in \mathbb{R}: 2<x<6\}$ denotes the set of all real numbers which are greater than 2 and less than 6.
This set is also frequently denoted as $(2,6)$.
The set $\{x \in \mathbb{R}: 2 \leq x \leq 6\}$ is denoted as $[2,6]$
The set $\{x \in \mathbb{R}: 2<x \leq 6\}$ is denoted as $(2,6]$ etc.
The set $\{x \in \mathbb{R}: 2<x\}$ is denoted as $(2, \infty)$.
These types of sets are known as intervals.
An interval can be described as the set of real numbers between two numbers either including or excluding one or both of them. In mathematical terms, this looks as follows. For any two real numbers $a<b$, we may consider four different intervals:

- $(a, b)=\{x \in \mathbb{R} \mid a<x<b\}$, called an open interval;
- $[a, b)=\{x \in \mathbb{R} \mid a \leq x<b\}$, a half-closed interval;
- $(a, b]=\{x \in \mathbb{R} \mid a<x \leq b\}$, the other half-closed interval;
- $[a, b]=\{x \in \mathbb{R} \mid a \leq x \leq b\}$, a closed interval.

In the following examples we give a visual impression of these four types of intervals.

## Example 0.3

We mark the included end-points by a filled dot in our pictures and excluded end-points by an unfilled dot:

The set $[2,8]$


The set ( 2,8 )


The set $(2,8]$


The set $[2,8)$


In addition to these finite intervals, we also need infinite intervals. These are intervals as before with $a$ replaced by the symbol $-\infty$ or $b$ replaced by the symbol $\infty$ or both. Because $\infty$ and $-\infty$ are not real numbers, they cannot be included in these intervals. So, we can have open and half-closed infinite intervals. There are five different versions of infinite intervals. With real numbers $a, b$ these are

$$
(-\infty, b), \quad(-\infty, b], \quad(a, \infty), \quad(a, \infty] \quad \text { and } \quad(-\infty, \infty)=\mathbb{R}
$$

For example, we have

- $[0, \infty)=\{x \in \mathbb{R} \mid 0 \leq x\}$ is the set of all non-negative real numbers.
- $(-\infty, \sqrt{3})=\{x \in \mathbb{R} \mid x<\sqrt{3}\}$ comprises all real numbers which are smaller than $\sqrt{3}$.

Unfortunately the symbol $(a, b)$ also denotes an ordered pair. We have to rely on the context to indicate what meaning is to be ascribed to $(a, b)$.
If $(a, b)$ is an ordered pair then $(a, b) \neq(b, a)$ unless $a=b$.

## Example 0.4

The pair (first name, second name) is an ordered pair. If $(1,2)$ is a pair of co-ordinates then it is an ordered pair.


## Functions:

A function is a set of ordered pairs $(x, y)$ with the restriction that, if $\left(x, y_{1}\right)$ and $\left(x, y_{2}\right)$ are in the function then $y_{1}=y_{2}$. That is, each $x$ is paired with only one $y$.

Note, however, that we can have $\left(x_{1}, y\right)$ and $\left(x_{2}, y\right)$ in a function with $x_{1} \neq x_{2}$. That is, two different $x$ 's can be related to the same $y$.
The language used when speaking and writing about functions is often quite loose. A function is usually specified by referring to the rule which sets up the ordered pairs of the function. For example, we speak of

- 'the function $y=x^{2}$; '
- 'the function $f(x)=x^{2}$; '
- 'the function $x^{2}$. '

All these phrases designate the function $\left\{\left(x, x^{2}\right): x \in \mathbb{R}\right\}$.
Note that the second co-ordinate of an ordered pair is frequently denoted by the letter $y$ or by some expression such as $f(x)$ or $g(x)$ or $\sin (x)$ etc.
The thinking in this latter type of notation is that $f(x)$ etc. is the element that
is paired with $x$.
Another common notation is $x \mapsto f(x)$ for the function $f(x)$ etc. This kind of notation gives rises to phrases like ' $x$ is mapped to' or 'sent to $f(x)$ ' by the function, which means that the pair $(x, f(x))$ is in the function.

### 0.1.1 Graphs

If a function is made up of pairs of real numbers we can make a useful picture of the function using a Cartesian diagram and from now on we will assume that all the functions we deal with consist of ordered pairs of real numbers.

Using the usual rules we represent each element $(x, y)$ of a function by a point (visually a dot) in a Cartesian diagram:

This dot represents the element $(x, y) \uparrow$


A function is then represented by all the dots that correstpond to the ordered pairs of the function.
Such a picture is called the graph of the function.

The following is the graph of the function $x^{2}, \quad x \in[-2,2]$ :


The function $f(x)=x^{2}, \quad x \in(-4,4)$ has graph:


A function defined by a single formula can have a discontinuous graph:

## Example 0.5

Sketch the graph of the function $f(x)=[x]$.
(Note: $[x]$ is defined to be the integer part of $x$. e.g. $[2.5]=2,[-1.4]=$ $-1,[\sqrt{2}]=1$ etc.)


If a function has an infinite number of ordered pairs we can only sketch its graph.
We can plot a few points and use some analysis to determine other helpful facts about the graph.

## Example 0.6

Sketch the graph of the function $y=x^{2}-5 x+6$.

We can plot some points at any time but it is probably sensible to determine significant facts about where this graph crosses the axes.

- When $x=0$ we get $y=6$. That is the point $(0,6)$ is on the graph. This point is on the $y$-axis.
- $y=0$ when $x^{2}-5 x+6=0$ and we know that this happens when $x=2$ and $x=3$.

We can also note that, for large positive and large negative values of $x$ a quadratic $a x^{2}+b x+c$ will be positive if $a$ is positive and negative if $a$ is negative.

Thus we arrive at the sketch:


## Example 0.7

Sketch the graph of the function $y=x^{2}+x+1$.
When $x=0$ we get $y=1$. That is the point $(0,1)$ is on the graph. This point is on the $y$-axis.

$$
y=0 \Rightarrow x^{2}+x+1=0 \Rightarrow x=\frac{-1 \pm \sqrt{-3}}{2}
$$

Since $\sqrt{-3}$ is complex we see that there is no real value $x$ which makes $x^{2}+x+1$ equal to 0 .
We conclude that the graph of $y=x^{2}+x+1$ never crosses the $x$-axis and
therefore always lies above or below the $x$-axis. Since, as we saw above, the point $(0,1)$ is on the graph we conclude that the graph of $y=x^{2}+x+1$ always lies above the $x$-axis.
From the remarks in the example above we also note that $y=x^{2}+x+1$ becomes large and positive as $x$ becomes large positive and large negative. Finally, noting that $(1,3)$ and $(-1,1)$ are on the graph we get the sketch:


Frequently the rule defining a function can not be encapsulated in a single formula.

## Example 0.8

$$
f(x)= \begin{cases}-2, & \text { when } x<-1 \\ 2, & \text { when } x \geq-1\end{cases}
$$

is a function. The following is a sketch of its graph:


### 0.1.2 Domain of a function

The set of all the $x$ values of the ordered pairs $(x, y)$ of a function is called the domain of the function.
The domain may be mentioned explicitly but, if not, it is understood to be the largest set of real numbers for which the function definition makes sense.

## Example 0.9

- The function $y=\sqrt{x}$ has domain $[0, \infty)$ since the square root of a negative number is not real.
- The function $y=\frac{1}{x}$ has domain $\mathbb{R} \backslash\{0\}$ because we cannot divide by 0 .
- The function $f(x)=x^{2}, \quad x \in[-4,4]$ has domain $[-4,4]$, as specified, and has graph:

- The function $f(x)=x^{2}, \quad x \in(-4,4)$ has domain $(-4,4)$, as specified, and has graph:

- The function $f(x)=\left\{\begin{array}{ll}x^{2}, & x \in(-4,4) \\ 3, & x=-4 \\ -3, & x=4\end{array}\right.$ has domain $[-4,4]$ and graph:

0.1.2.1 Range of a function. The set of all the $y$ values of the ordered pairs $(x, y)$ of a function is called the range of the function.


### 0.1.3 Sequences

A sequence is a special kind of function. It is a function whose domain is a set of integers. Frequently the domain is $\mathbb{N}$.

Because they are of particular importance a special notation has evolved for sequences.
The sequence $\{(n, f(n)): n \in \mathbb{N}\}$ is more often referred to as:

- The sequence $\{f(n)\}_{n \in \mathbb{N}}$
- The sequence $\{f(n)\}$
- The sequence $f(1), f(2)$, $\qquad$ $f(n), \ldots \ldots$.
- The sequence $\left\{a_{n}\right\}_{n \in \mathbb{N}}$ where $a_{n}=f(n)$.
- The sequence $\left\{a_{n}\right\}$ where $a_{n}=f(n)$.
- The sequence $a_{1}, a_{2}$ $\qquad$ where $a_{n}=f(n)$.

A graph of a sequence will be "dotty" since the domain is a set of integers.

## Example:

The following is an attempt to sketch a graph of the sequence $\left\{\frac{1}{n}\right\}$.
(The small arrow at the right hand side is an indication that the graph continues on forever.)


